THE STABILITY OF SOLUTIONS OF TWO NONLINEAR DIFFERENTIAL EQUATIONS OF THE THIRD AND FOURTH ORDERS

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Pliss [1] investigated the stability of the zero solution of equation

$$\frac{d^3\xi}{dt^3} + f\left(\frac{d^2\xi}{dt^2}\right) + \frac{d\xi}{dt} + \xi = 0 \tag{1}$$

where the function $f(\eta)$ is continuous and satisfies the Lipschitz condition for all real η. It was also assumed that $f(0) = 0$. He showed that the zero solution of equation (1) is in general stable, provided that the function $f(\eta)$ is differentiable and $df/d\eta > 1$ for all η.

Here we first consider an equation analogous to (1), namely

$$\frac{d^3\xi}{dt^3} + f\left(\frac{d^2\xi}{dt^2}\right) + b\frac{d\xi}{dt} + a\xi = 0 \tag{2}$$

Here a and b denote constants, and as in equation (1) the function $f(\eta)$ is continuous and satisfies the Lipschitz condition, and in addition $f(0) = 0$.

Using the assumed differentiability property of $f(\eta)$, we will give a method somewhat different from that in paper [1] for the construction of the Liapunov function for equation (1). We will then show that by the same method the Liapunov function can be constructed for equation

$$\frac{d^4\xi}{dt^4} + f\left(\frac{d^3\xi}{dt^3}\right) + c\frac{d^2\xi}{dt^2} + b\frac{d\xi}{dt} + a\xi = 0 \tag{3}$$

Theorem 1. If $a > 0$, $b > 0$, the function $f(\eta)$ is differentiable and

$$\frac{df}{d\eta} > \frac{a}{b} \qquad \text{for all } \eta\,; \tag{4}$$

then the zero solution of equation (2) is asymptotically stable for arbitrary initial perturbations.

To prove this theorem, replace equation (2) by the equivalent system

$$\frac{dx}{dt} = y, \qquad \frac{dy}{dt} = z - f'(x)\,y, \qquad \frac{dz}{dt} = -by - ax \tag{5}$$

and consider the function

$$2V = 2a\int_0^x f'(x)\,x\,dx + 2axy + by^2 + z^2 \tag{6}$$

This function is obviously positive definite, and its derivative with respect to time, owing to system (5), is always negative:

$$\frac{dV}{dt} = -[bf'(x) - a]\,y^2 \qquad \left(\frac{dV}{dt} < 0 \text{ for } y \neq 0, \quad \frac{dV}{dt} = 0 \text{ for } y = 0\right) \tag{7}$$

Following [2], denote by M the set of points of the xz-plane. It is easy to see that it does not contain whole trajectories except the origin of the coordinate system. If

$$\lim_{|x|\to\infty} J(x) = \infty \qquad \left(J(x) = 2a\int_0^x f'(x)\,x\,dx - \frac{a^2}{b}x^2\right) \tag{8}$$

now holds good, then the function $V(x, y, z)$ is infinitely large (all its level surfaces are closed) and the proposition is proved (see Theorem 4 of [2]).

If, however,

$$\int_0^{\pm\infty}\left[f'(x) - \frac{a}{b}\right]x\,dx$$

converges, then among the level surfaces of the function $V(x, y, z)$, there will necessarily be open surfaces. In that case, repeating the relevant arguments of Shimanov [3], it can be shown that in general there is stability.

In fact, consider the region

$$V(x, y, z) \leqslant l, \qquad |x| \leqslant N \tag{9}$$

where $l > 0$ and $N > 0$ are certain constants. This region is bounded, since for $|x| \leqslant N$ the boundedness of the y and z coordinates of the points of region (9) follows from the first inequality $V \leqslant l$.

Let $P(x_0, y_0, z_0)$ be an arbitrary point of the phase space. Consider an arbitrary trajectory γ of the system (5) given by equations $x = x(t)$ $y = y(t)$, $z = z(t)$ and issuing from the point P. Further select the constants l and N so large that the point P is inside the region (9), i.e. the inequalities

$$V(x_0, y_0, z_0) < l \qquad |x_0| < N$$

are satisfied.

Then for $t > 0$ all points of the trajectory γ will remain inside the region (9), i.e. the inequalities

$$V(x(t), y(t), z(t)) < l, \qquad |x(t)| < N \qquad \text{for } t > 0 \tag{10}$$

will hold good.

In fact, if some point of the trajectory γ leaves the region (9), then there will be a value of $t = T$ for which the point $[x(T), z(T), z(T)]$ will lie on the boundary of the region (9). Then one of the inequalities (10) (or both simultaneously) will become equality. The first inequality, however, cannot become an equality owing to condition (7), according to which

$$V(x(T), y(T), z(T)) \leqslant V(x_0, y_0, z_0) < l$$

The second inequality can become an equality only if, being the boundary of (9), the set

$$V(x, y, z) \leqslant l, \qquad |x| = N \tag{11}$$

is not empty. But in that case the constant N can be chosen so large that for the points of (11) condition

$$yN \operatorname{sgn} x = \frac{dx}{dt} N \operatorname{sgn} x < 0 \tag{12}$$

holds good. This is because the y coordinate of the points of (11) satisfies the inequality

$$-aN \operatorname{sgn} x - F(N, z) \leqslant by \leqslant -aN \operatorname{sng} x + F(N, z)$$

where

$$F(N, z) = \left| \sqrt{2bl - bJ(N) - bz^2} \right|$$

(the expression under the square root sign in $F(N, z)$ assumes positive values not exceeding $2bl$.

In accordance with (12), the integral curve on the set (11) crosses this set in the direction of decreasing x for $N \operatorname{sgn} x > 0$,

Thus we have shown that for $t > 0$ all trajectories of system (5) are inside the bounded region (9). Together with the above deductions as to V and its derivative with respect to time, this guarantees the general asymptotic stability of the zero solution of system (5) and consequently of equation (2). Thus the theorem is proved.

Now consider the equation

$$\frac{d^4\xi}{dt^4} + f\left(\frac{d^3\xi}{dt^3}\right) + c\,\frac{d^2\xi}{dt^2} + b\,\frac{d\xi}{dt} + a\xi = 0 \tag{13}$$

where the function $f(\eta)$ is continuous and satisfies the Lipschitz condition and $f(0) = 0$.

Theorem 2. If $a > 0$, $b > 0$ and the derivative $df/d\eta$ of the function $f(\eta)$ is such that

$$\frac{df}{d\eta} > 0, \qquad bc\,\frac{df}{d\eta} - b^2 - a\left(\frac{df}{d\eta}\right)^2 > 0 \tag{14}$$

holds good for all values of the argument, then the zero solution of equation (13) is asymptotically stable for arbitrary initial perturbations.

Proof. Replace equation (13) by the equivalent system

$$\frac{dx}{dt} = y, \qquad \frac{dy}{dt} = z, \quad \frac{dz}{dt} = u - f'(y)\,z, \quad \frac{du}{dt} = -cz - by - ax \tag{15}$$

and consider the function

$$2V = (b^2 + ac)\,x^2 + 2bcxy + (c^2 - 2a)\,y^2 + 4axz + 2byz + cz^2 + 2bxu + {} + 2cyu + 2u^2 + 2b\int_0^y f'(y)\,y\,dy \tag{16}$$

This function is positive definite and infinitely large. In fact, we have

$$2V = (b^2 + ac)\,x^2 + 2bcxy + \left(c^2 - 2a + \frac{b^2}{c}\right)y^2 + 4axz + 2byz + {} + cz^2 + 2bxu + 2cyu + 2u^2 + 2b\int_0^y \left[f'(y) - \frac{b}{c}\right] y\,dy \tag{17}$$

where

$$\int_0^y \left[f'(y) - \frac{b}{c}\right] y\,dy > 0 \quad \text{for } y \neq 0$$

This inequality holds good because by (14) $f'(y) > b/c$ and the quadratic form is positive definite. This assertion can easily be verified by the Sylvester criterion if we allow for the inequality $c^2 - 4a > 0$, which also follows from (14).

Since every positive definite form is an infinitely large function, our assertion as to the function, $V(x, y, z, u)$ is completely proved.

The derivative with respect to time of the function V, allowing for system of equations (15), is

$$\frac{dV}{dt} = -abx^2 - 2af'(y)xz - cf'(y)z^2 + bz^2$$

It is easy to see that

$$\frac{dV}{dt} < 0 \quad \text{for } x \neq 0,\ z \neq 0; \qquad \frac{dV}{dt} = 0 \quad \text{for } x = 0,\ z = 0 \tag{18}$$

The yu-plane obviously does not contain whole trajectories, except the origin of the coordinates. Thus the zero solution of equation (13) is asymptotically stable for arbitrary initial perturbations [2].

The definiteness of the sign of the function V, and the fact that its derivative with respect to time is a function of fixed sign, also follow from the general propositions on the Liapunov functions of the type considered, as established in paper [5].

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